## Homological Algebra Seminar Week 5

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## 2 Derived functors

Throughout this whole chapter,  $\mathcal{A}$  and  $\mathcal{B}$  will denote two arbitrary abelian categories.

## 2.1 $\delta$ -functors

**Definition 2.1.** A homological (resp. cohomological)  $\delta$ -functor between  $\mathcal{A}$  and  $\mathcal{B}$  is a collection of additive functors  $\{T_n : \mathcal{A} \to \mathcal{B}\}_{n \geq 0}$  (resp.  $\{T^n : \mathcal{A} \to \mathcal{B}\}_{n \geq 0}$ ) together with a collection of morphisms  $\{\delta_n : T_n(C) \to T_{n-1}(A)\}$  (resp.  $\{\delta^n : T^n(C) \to T^{n+1}(A)\}$ ) defined for every short exact sequence  $0 \to A \to B \to C \to 0$  in  $\mathcal{A}$ , such that the two following conditions hold:

1. For any short exact sequence  $0 \to A \to B \to C \to 0$  in  $\mathcal{A}$ , we have a long exact sequence

$$\cdots \to T_{n+1}(C) \stackrel{\delta_{n+1}}{\to} T_n(A) \to T_n(B) \to T_n(C) \stackrel{\delta_n}{\to} T_{n-1}(A) \to \cdots$$

(resp. 
$$\cdots \to T^{n-1}(C) \xrightarrow{\delta^{n-1}} T^n(A) \to T^n(B) \to T^n(C) \xrightarrow{\delta^n} T^{n+1}(A) \to \cdots$$
)

2. For every morphism of short exact sequences

we have a commutative diagram for every  $n \in \mathbb{Z}$ :

$$T_n(C') \xrightarrow{\delta_n} T_{n-1}(A')$$
 resp.  $T^n(C') \xrightarrow{\delta^n} T^{n+1}(A')$ 

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$T_n(C) \xrightarrow{\delta_n} T_{n-1}(A) \qquad \qquad T^n(C) \xrightarrow{\delta^n} T^{n+1}(A)$$

**Remark 2.2.** We make the convention that  $T_n$  (resp.  $T^n$ ) is 0 for any n < 0. In particular by Definition 2.1.1,  $T_0$  is right-exact (resp.  $T^0$  is left-exact).

- **Example 2.3.** 1. Homology gives a homological  $\delta$ -functor  $\operatorname{Ch}_{\geq 0}\mathcal{A} \to \mathcal{A}$ , and cohomology gives a cohomological one  $\operatorname{Ch}^{\geq 0}\mathcal{A} \to \mathcal{A}$ .
  - 2. Let A be an abelian group, and  $p \in \mathbb{Z}$ . We define

$$T_0(A) := A/pA, T_1(A) := pA := \{a \in A | pa = 0\}, T_n = 0, \forall n \ge 2$$

and we want then to define  $\delta_1$  to get a homological  $\delta$ -functor  $\mathbf{Ab} \to \mathbf{Ab}$  (or a cohomological one by  $T^0 := T_1, T^1 := T_0, \delta^0 := \delta_1$ ).

For this, let  $0 \to A \to B \to C \to 0$  be a short exact sequence, and consider the following diagram:

By the snake lemma we have an exact sequence

$$0 \rightarrow {}_{p}A \rightarrow {}_{p}B \rightarrow {}_{p}C \xrightarrow{\delta} A/{}_{p}A \rightarrow B/{}_{p}B \rightarrow C/{}_{p}C \rightarrow 0$$

and this gives us  $\delta_1 := \delta$ .

3. We can generalize the previous example to the category of R-modules for some ring R. To do that, let  $r \in R, M \in R$ -mod, and define

$$T_0(M) := M /_{rM}, T_1(M) := {_rM}$$

to get a homological  $\delta$ -functor R-mod  $\rightarrow$  Ab.

4. In the same setting as in point 3., we can also define

$$T_n(M) := \operatorname{Tor}_n^R \left( R / (r), M \right), n \ge 0$$

and we will see later in the chapter why this is a homological  $\delta$ -functor.

**Definition 2.4.** Let  $S_{\bullet}, T_{\bullet}$  be two homological  $\delta$ -functors (resp.  $S^{\bullet}, T^{\bullet}$  two cohomological  $\delta$ -functors). A morphism  $S_{\bullet} \to T_{\bullet}$  (resp.  $S^{\bullet} \to T^{\bullet}$ ) is a collection of natural transformations  $\alpha_n : S_n \to T_n$  (resp.  $\alpha^n : S^n \to T^n$ ) that commutes with  $\delta$ , i.e. such that for any short exact sequence  $0 \to A \to B \to C \to 0$ , the following diagram commutes:

$$\cdots \to S_{n+1}(C) \xrightarrow{\delta_{n+1}^S} S_n(A) \to S_n(B) \to S_n(C) \xrightarrow{\delta_n^S} S_{n-1}(A) \to \cdots$$

$$(\alpha_{n+1})_C \downarrow \qquad (\alpha_n)_A \downarrow \qquad (\alpha_n)_B \downarrow \qquad (\alpha_n)_C \downarrow \qquad (\alpha_{n-1})_A \downarrow$$

$$\cdots \to T_{n+1}(C)_{\delta_{n+1}^{\overrightarrow{T}}} T_n(A) \to T_n(B) \to T_n(C) \xrightarrow{\delta_n^T} T_{n-1}(A) \to \cdots$$

resp.

$$\cdots \longrightarrow S^{n-1}(C) \xrightarrow{\delta_S^{n-1}} S^n(A) \longrightarrow S^n(B) \longrightarrow S^n(C) \xrightarrow{\delta_S^n} S^{n+1}(A) \longrightarrow \cdots$$

$$(\alpha^{n-1})_C \downarrow \qquad (\alpha^n)_A \downarrow \qquad (\alpha^n)_B \downarrow \qquad (\alpha^n)_C \downarrow \qquad (\alpha^{n+1})_A \downarrow$$

$$\cdots \longrightarrow T^{n-1}(C) \xrightarrow{\delta_T^{n+1}} T^n(A) \longrightarrow T^n(B) \longrightarrow T^n(C) \xrightarrow{\delta_T^n} T^{n+1}(A) \longrightarrow \cdots$$

**Definition 2.5.** A homological  $\delta$ -functor  $T_{\bullet}$  (resp. cohomological  $\delta$ -functor  $T^{\bullet}$ ) is universal if for any other  $\delta$ -functor  $S_{\bullet}$  and any natural transformation  $f_0: S_0 \to T_0$  (resp. any  $S^{\bullet}$  and any  $f^0: T^0 \to S^0$ ), there exists a unique morphism  $\{f_n: S_n \to T_n\}$  extending  $f_0$  (resp. there exists a unique morphism  $\{f^n: T^n \to S^n\}$  extending  $f^0$ ).

**Example 2.6.** We will see later that homology  $H_*: \operatorname{Ch}_{\geq 0} \mathcal{A} \to \mathcal{A}$  and cohomology  $H^*: \operatorname{Ch}^{\geq 0} \mathcal{A} \to \mathcal{A}$  are universal.

## 2.2 Projective resolutions

**Definition 2.7.** An object P in an abelian category A is *projective* if it satisfies the following universal property:

 $\forall B \xrightarrow{g} C$  epimorphism,  $P \xrightarrow{\gamma} C, \exists P \xrightarrow{\beta} B$  such that the following diagram commutes:

$$B \xrightarrow{\exists \beta} P$$

$$\downarrow^{\gamma}$$

$$\downarrow^{\gamma}$$

In other words, the morphism  $\operatorname{Hom}_{\mathcal{A}}(P,B) \to \operatorname{Hom}_{\mathcal{A}}(P,C)$  induced by g is surjective.

**Example 2.8.** Free *R*-modules are projective, as you can lift the image by  $\gamma$  of a basis of *P* by *q* to get  $\beta$ .

**Proposition 2.9.** An R-module is projective if and only if it's a direct summand of a free module.

*Proof.* " $\Leftarrow$ " This is clear by the universal property of coproduct and the fact that free R-modules are projective.

" $\Rightarrow$ " Let A be a projective module, and F(A) be the free R-module with basis  $\{e_a\}_{a\in A}$ . Note that F(A) is equipped with a projection  $\pi: F(A) \to A$ . Now, by the universal property of projective modules, we have a morphism  $i: A \to F(A)$  such that  $\pi i = id_A$ , so that

$$0 \to A \xrightarrow{i} F(A) \to F(A)/A \to 0$$

is split, and that A is a direct summand of F(A).

- **Example 2.10.** 1. Let  $R := R_1 \times R_2$  a product of two rings, and  $P = R_1 \times 0$  an R-module. P is projective as it is a direct summand of R, but it is not free as  $(0,1) \cdot P = 0$ .
  - 2. Let F be a field,  $R = M_n(F), n > 1$ .  $V := F^n$  is a projective R-module, as  $R = V^{\oplus n}$ , but V is not free; indeed, if it was, its dimension as a F-vector space would be  $dn^2$  for some  $d \ge 0$ , but  $\dim_F V = n \ne dn^2$ .

**Definition 2.11.** We say that an abelian category  $\mathcal{A}$  has enough projectives if for every object A of  $\mathcal{A}$ , there is a projective module P and an epimorphism  $P \to A$ .

Remark 2.12. The category of finite abelian groups is an abelian category with no non-zero projective object.

**Lemma 2.13.** Let  $M \in \mathcal{A}$ . M is projective if and only if  $Hom_{\mathcal{A}}(M, -)$  is exact.

*Proof.* Let  $0 \to A \to B \to C \to 0$  be a short exact sequence in  $\mathcal{A}$ . We already know that  $\operatorname{Hom}_{\mathcal{A}}(M,-)$  is left exact, so saying that it is exact is equivalent to saying that

$$\operatorname{Hom}_{\mathcal{A}}(M,B) \to \operatorname{Hom}_{\mathcal{A}}(M,C)$$

is surjective. But this is exactly the definition that we gave (Definition 2.7), so we are done.  $\hfill\Box$ 

**Definition 2.14.** A *left resolution* of  $M \in \mathcal{A}$  is a chain complex  $P_{\bullet}$  bounded below by 0, such that there is a map  $P_0 \to M$  making the following sequence exact:

$$\cdots \rightarrow P_3 \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

If every  $P_i$  is projective, then we say that  $P_{\bullet}$  is a projective resolution.

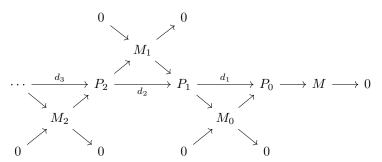
**Lemma 2.15.** Let A be an abelian category with enough projectives. Then every object M of A has a projective resolution.

*Proof.* We will construct  $P_i$  by induction. First, as  $\mathcal{A}$  has enough projectives, there is a projective module and an epimorphism  $P_0 \twoheadrightarrow M \to 0$ . We moreover define  $M_0 := \ker(P_0 \twoheadrightarrow M)$ .

Inductively, having defined  $P_k, M_k, \forall k \leq i-1$ , we define  $P_i$  to be the projective module with an epimorphism  $P_i \to M_{i-1} \to 0$ , and  $M_i$  to be the kernel of this morphism, namely  $M_i := \ker(P_i \to M_{i-1})$ .

Writing  $d_i$  for the composite  $P_i \rightarrow M_{i-1} \rightarrow P_{i-1}$ , we have a commutative

diagram



where every  $0 \to M_i \to P_i \to M_{i-1} \to 0$  is exact by definition of  $M_i$  and  $P_i$ . But this exactness precisely gives us that

$$d_i(P_i) = M_{i-1} = \ker(d_{i-1}),$$

which shows that  $P_{\bullet}$  is a left resolution of M.

**Theorem 2.16** (Comparison Theorem). Let  $f: M \to N$  be a map in  $\mathcal{A}$ . Moreover let  $P_{\bullet} \to M$  be a projective resolution, and  $Q_{\bullet} \to N$  any left resolution. Then there is a chain map  $f_{\bullet}: P_{\bullet} \to Q_{\bullet}$  extending f, i.e. such that the following diagram commutes:

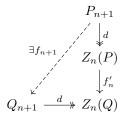
$$\begin{array}{cccc}
\cdots & \longrightarrow P_1 & \longrightarrow P_0 & \longrightarrow M & \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
f_1 \downarrow & & \downarrow & & \downarrow \\
\downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow Q_1 & \longrightarrow Q_0 & \longrightarrow N & \longrightarrow 0
\end{array}$$

This map is unique up to homotopy.

*Proof.* We do the proof by constructing  $f_n$  inductively, where  $n \geq -1$ . For the base case, we define  $f_{-1} := f, P_{-1} := M, Q_{-1} := N$ , and moreover we denote by  $d_0$  the maps  $P_0 \to P_{-1}$  and  $Q_0 \to Q_{-1}$ , so that when we talk about  $P_{\bullet}$  or  $Q_{\bullet}$ , we really mean the exact sequences extended by the term  $P_{-1}$  or  $Q_{-1}$ .

Now suppose we have constructed  $f_k, \forall k \leq n$ . By the equality  $f_{n-1}d = df_n$ , we have an induced map  $f'_n: Z_n(P) \to Z_n(Q)$ .

But  $Z_n(P) = B_n(P)$  by exactness of  $P_{\bullet}$ , so  $d: P_{n+1} \to P_n$  factorizes by  $d: P_{n+1} \to Z_n(P)$  (and similarly for Q). We therefore have, by projectivity of  $P_{n+1}$ , the existence of a map  $f_{n+1}$  such that the following diagram commutes:



Moreover,  $df_{n+1} = f'_n d = f_n d$  so this indeed is a chain map.

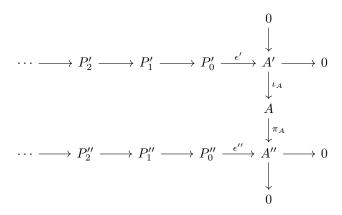
To see uniqueness up to homotopy, we let  $g_{\bullet}$  be another candidate to extend f. We want to construct  $\{s_n: P_n \to Q_{n+1}\}_{n \geq -1}$  such that h:=f-g=sd+ds. First define  $s_{-1}=0$ . Note that  $d_0h_0=h_{-1}d_0=(f-f)d_0=0$ , so that  $h_0$  factors by  $h_0: P_0 \to Z_0(Q)=B_0(Q)$ . By projectivity of  $P_0$ ,  $h_0$  lifts to a map  $s_0: P_0 \to Q_1$  such that  $h_0=ds_0=s_{-1}d+ds_0$ , and we have the base case.

Assume now by induction that we are given maps  $s_i, \forall i < n$  such that  $ds_i = h_i - s_{i-1}d$ . In particular, the map  $h_n - s_{n-1}d : P_n \to Q_n$  satisfies

$$d(h_n - s_{n-1}d) = dh_n - (h_{n-1} - ds_{n-2})d = dh - hd + sdd = 0$$

and this maps factors through  $Z_n(Q)$ . As before, that means it lifts to a map  $s_n: P_n \to Q_{n+1}$  with  $ds_n = h_n - s_{n-1}d$ , and we have our homotopy by induction.  $\square$ 

Lemma 2.17 (Horseshoe lemma). Given a commutative diagram



where the column is exact and the rows are projective resolutions, and defining  $P_n := P'_n \oplus P''_n$ , we have that  $P_{\bullet}$  is a projective resolution of A, and that the right hand column of the diagram lifts to an exact sequence of chain complexes

$$0 \to P' \stackrel{\iota}{\to} P \stackrel{\pi}{\to} P'' \to 0$$

where  $\iota_n: P'_n \to P_n, \pi_n: P_n \to P''_n$  are the natural inclusion and projection.

*Proof.* See exercise sheet 5.