Homological Algebra Seminar Week 5

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2 Derived functors

Throughout this whole chapter, $\mathcal{A}$ and $\mathcal{B}$ will denote two arbitrary abelian categories.

2.1 δ-functors

Definition 2.1. A *homological* (resp. *cohomological*) *δ-functor* between $\mathcal{A}$ and $\mathcal{B}$ is a collection of additive functors $\{T_n : \mathcal{A} \to \mathcal{B}\}_{n\geq 0}$ (resp. $\{T^n : \mathcal{A} \to \mathcal{B}\}_{n\geq 0}$) together with a collection of morphisms $\{\delta_n : T_n(C) \to T_{n-1}(A)\}$ (resp. $\{\delta^n : T^n(C) \to T^{n+1}(A)\}$) defined for every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, such that the two following conditions hold:

1. For any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, we have a long exact sequence

$$\cdots \to T_{n+1}(C) \xrightarrow{\delta_{n+1}} T_n(A) \longrightarrow T_n(B) \longrightarrow T_n(C) \xrightarrow{\delta_n} T_{n-1}(A) \longrightarrow \cdots$$

(resp. $$\cdots \to T^{n-1}(C) \xrightarrow{\delta^{n-1}} T^n(A) \to T^n(B) \to T^n(C) \xrightarrow{\delta^n} T^{n+1}(A) \to \cdots)$$

2. For every morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

we have a commutative diagram for every $n \in \mathbb{Z}$:

$$\begin{array}{ccc} T_n(C') & \xrightarrow{\delta_n} & T_{n-1}(A') \\ \downarrow & & \downarrow \\ T_n(C) & \xrightarrow{\delta_n} & T_{n-1}(A) \end{array} \qquad \text{resp.} \qquad \begin{array}{ccc} T^n(C') & \xrightarrow{\delta^n} & T^{n+1}(A') \\ \downarrow & & \downarrow \\ T^n(C) & \xrightarrow[\delta^n]{} & T^{n+1}(A) \end{array}$$

Remark 2.2. We make the convention that T_n (resp. T^n) is 0 for any $n < 0$. In particular by Definition 2.1.1, T_0 is right-exact (resp. T^0 is left-exact).

Example 2.3.

1. Homology gives a homological δ-functor $\mathrm{Ch}_{\geq 0}\mathcal{A} \to \mathcal{A}$, and cohomology gives a cohomological one $\mathrm{Ch}^{\geq 0}\mathcal{A} \to \mathcal{A}$.

2. Let A be an abelian group, and $p \in \mathbb{Z}$. We define

$$T_0(A) := {}^{A}/_{pA}, T_1(A) := {}_pA := \{a \in A | pa = 0\}, T_n = 0, \forall n \geq 2$$

and we want then to define δ_1 to get a homological δ-functor $\mathbf{Ab} \to \mathbf{Ab}$ (or a cohomological one by $T^0 := T_1, T^1 := T_0, \delta^0 := \delta_1$).
For this, let $0 \to A \to B \to C \to 0$ be a short exact sequence, and consider the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\cdot p"'] & B \arrow[r] \arrow[d, "\cdot p"'] & C \arrow[r] \arrow[d, "\cdot p"'] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

By the snake lemma we have an exact sequence

$$0 \to {}_pA \to {}_pB \to {}_pC \xrightarrow{\delta} {}^{A}/_{pA} \to {}^{B}/_{pB} \to {}^{C}/_{pC} \to 0$$

and this gives us $\delta_1 := \delta$.

3. We can generalize the previous example to the category of R-modules for some ring R. To do that, let $r \in R, M \in R$-**mod**, and define

$$T_0(M) := {}^{M}/_{rM}, T_1(M) := {}_rM$$

to get a homological δ-functor R-**mod** $\to$ **Ab**.

4. In the same setting as in point 3., we can also define

$$T_n(M) := \mathrm{Tor}_n^R\left({}^{R}/_{(r)}, M\right), n \geq 0$$

and we will see later in the chapter why this is a homological δ-functor.

Definition 2.4. Let $S_\bullet, T_\bullet$ be two homological δ-functors (resp. $S^\bullet, T^\bullet$ two cohomological δ-functors). A *morphism* $S_\bullet \to T_\bullet$ (resp. $S^\bullet \to T^\bullet$) is a collection of natural transformations $\alpha_n : S_n \to T_n$ (resp. $\alpha^n : S^n \to T^n$) that commutes with δ, i.e. such that for any short exact sequence $0 \to A \to B \to C \to 0$, the following diagram commutes:

$$\begin{tikzcd} \cdots \arrow[r] & S_{n+1}(C) \arrow[r, "\delta_{n+1}^S"] \arrow[d, "(\alpha_{n+1})_C"'] & S_n(A) \arrow[r] \arrow[d, "(\alpha_n)_A"'] & S_n(B) \arrow[r] \arrow[d, "(\alpha_n)_B"'] & S_n(C) \arrow[r, "\delta_n^S"] \arrow[d, "(\alpha_n)_C"'] & S_{n-1}(A) \arrow[r] \arrow[d, "(\alpha_{n-1})_A"'] & \cdots \\ \cdots \arrow[r] & T_{n+1}(C) \arrow[r, "\delta_{n+1}^T"'] & T_n(A) \arrow[r] & T_n(B) \arrow[r] & T_n(C) \arrow[r, "\delta_n^T"'] & T_{n-1}(A) \arrow[r] & \cdots \end{tikzcd}$$

resp.

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & S^{n-1}(C) & \xrightarrow{\delta_S^{n-1}} & S^n(A) & \to & S^n(B) & \to & S^n(C) & \xrightarrow{\delta_S^n} & S^{n+1}(A) & \to \cdots \\
& {\scriptstyle (\alpha^{n-1})_C}\downarrow & & {\scriptstyle (\alpha^n)_A}\downarrow & & {\scriptstyle (\alpha^n)_B}\downarrow & & {\scriptstyle (\alpha^n)_C}\downarrow & & {\scriptstyle (\alpha^{n+1})_A}\downarrow & \\
\cdots \longrightarrow & T^{n-1}(C) & \xrightarrow[\delta_T^{n+1}]{} & T^n(A) & \to & T^n(B) & \to & T^n(C) & \xrightarrow[\delta_T^n]{} & T^{n+1}(A) & \to \cdots
\end{array}$$

Definition 2.5. A homological δ-functor $T_\bullet$ (resp. cohomological δ-functor $T^\bullet$) is *universal* if for any other δ-functor $S_\bullet$ and any natural transformation $f_0 : S_0 \to T_0$ (resp. any $S^\bullet$ and any $f^0 : T^0 \to S^0$), there exists a unique morphism $\{f_n : S_n \to T_n\}$ extending f_0 (resp. there exists a unique morphism $\{f^n : T^n \to S^n\}$ extending f^0).

Example 2.6. We will see later that homology $H_* : \mathrm{Ch}_{\geq 0}\mathcal{A} \to \mathcal{A}$ and cohomology $H^* : \mathrm{Ch}^{\geq 0}\mathcal{A} \to \mathcal{A}$ are universal.

2.2 Projective resolutions

Definition 2.7. An object P in an abelian category $\mathcal{A}$ is *projective* if it satisfies the following universal property:
$\forall B \overset{g}{\twoheadrightarrow} C$ epimorphism, $P \overset{\gamma}{\to} C, \exists P \overset{\beta}{\to} B$ such that the following diagram commutes:

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "\exists\beta"'] \arrow[d, "\gamma"] \\
B \arrow[r, two heads, "g"] & C
\end{tikzcd}$$

In other words, the morphism $\mathrm{Hom}_{\mathcal{A}}(P, B) \to \mathrm{Hom}_{\mathcal{A}}(P, C)$ induced by g is surjective.

Example 2.8. Free R-modules are projective, as you can lift the image by γ of a basis of P by g to get β.

Proposition 2.9. *An R-module is projective if and only if it's a direct summand of a free module.*

Proof. "$\Leftarrow$" This is clear by the universal property of coproduct and the fact that free R-modules are projective.

"$\Rightarrow$" Let A be a projective module, and $F(A)$ be the free R-module with basis $\{e_a\}_{a \in A}$. Note that $F(A)$ is equipped with a projection $\pi : F(A) \to A$.
Now, by the universal property of projective modules, we have a morphism $i : A \to F(A)$ such that $\pi i = id_A$, so that

$$0 \to A \overset{i}{\to} F(A) \to {F(A)}/{A} \to 0$$

is split, and that A is a direct summand of $F(A)$. $\square$

Example 2.10. 1. Let $R := R_1 \times R_2$ a product of two rings, and $P = R_1 \times 0$ an R-module. P is projective as it is a direct summand of R, but it is not free as $(0,1) \cdot P = 0$.

2. Let F be a field, $R = M_n(F), n > 1$. $V := F^n$ is a projective R-module, as $R = V^{\oplus n}$, but V is not free; indeed, if it was, its dimension as a F-vector space would be dn^2 for some $d \geq 0$, but $\dim_F V = n \neq dn^2$.

Definition 2.11. We say that an abelian category $\mathcal{A}$ *has enough projectives* if for every object A of $\mathcal{A}$, there is a projective module P and an epimorphism $P \twoheadrightarrow A$.

Remark 2.12. The category of finite abelian groups is an abelian category with no non-zero projective object.

Lemma 2.13. *Let $M \in \mathcal{A}$. M is projective if and only if $Hom_{\mathcal{A}}(M, -)$ is exact.*

Proof. Let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathcal{A}$.
We already know that $\mathrm{Hom}_{\mathcal{A}}(M, -)$ is left exact, so saying that it is exact is equivalent to saying that

$$\mathrm{Hom}_{\mathcal{A}}(M, B) \to \mathrm{Hom}_{\mathcal{A}}(M, C)$$

is surjective. But this is exactly the definition that we gave (Definition 2.7), so we are done. $\square$

Definition 2.14. A *left resolution* of $M \in \mathcal{A}$ is a chain complex $P_\bullet$ bounded below by 0, such that there is a map $P_0 \to M$ making the following sequence exact:

$$\cdots \to P_3 \to P_2 \to P_1 \to P_0 \to M \to 0$$

If every P_i is projective, then we say that $P_\bullet$ is a projective resolution.

Lemma 2.15. *Let $\mathcal{A}$ be an abelian category with enough projectives. Then every object M of $\mathcal{A}$ has a projective resolution.*

Proof. We will construct P_i by induction. First, as $\mathcal{A}$ has enough projectives, there is a projective module and an epimorphism $P_0 \twoheadrightarrow M \to 0$. We moreover define $M_0 := \ker(P_0 \twoheadrightarrow M)$.

Inductively, having defined $P_k, M_k, \forall k \leq i-1$, we define P_i to be the projective module with an epimorphism $P_i \twoheadrightarrow M_{i-1} \to 0$, and M_i to be the kernel of this morphism, namely $M_i := \ker(P_i \twoheadrightarrow M_{i-1})$.
Writing d_i for the composite $P_i \to M_{i-1} \to P_{i-1}$, we have a commutative

diagram

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & & & \\
 & & & M_1 \arrow[ur] \arrow[dr] & & & & & \\
\cdots \arrow[rr, "d_3"] \arrow[dr] & & P_2 \arrow[ur] \arrow[rr, "d_2"'] & & P_1 \arrow[rr, "d_1"] \arrow[dr] & & P_0 \arrow[r] & M \arrow[r] & 0 \\
 & M_2 \arrow[ur] \arrow[dr] & & & & M_0 \arrow[ur] \arrow[dr] & & & \\
0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

where every $0 \to M_i \to P_i \to M_{i-1} \to 0$ is exact by definition of M_i and P_i.
But this exactness precisely gives us that

$$d_i(P_i) = M_{i-1} = \ker(d_{i-1}),$$

which shows that $P_\bullet$ is a left resolution of M. $\square$

Theorem 2.16 (Comparison Theorem)**.** *Let $f : M \to N$ be a map in $\mathcal{A}$. Moreover let $P_\bullet \to M$ be a projective resolution, and $Q_\bullet \to N$ any left resolution. Then there is a chain map $f_\bullet : P_\bullet \to Q_\bullet$ extending f, i.e. such that the following diagram commutes:*

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "f_1"'] & P_0 \arrow[r] \arrow[d, "f_0"'] & M \arrow[r] \arrow[d, "f"'] & 0 \\
\cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

This map is unique up to homotopy.

Proof. We do the proof by constructing f_n inductively, where $n \geq -1$.
For the base case, we define $f_{-1} := f, P_{-1} := M, Q_{-1} := N$, and moreover we denote by d_0 the maps $P_0 \to P_{-1}$ and $Q_0 \to Q_{-1}$, so that when we talk about $P_\bullet$ or $Q_\bullet$, we really mean the exact sequences extended by the term P_{-1} or Q_{-1}.

Now suppose we have constructed $f_k, \forall k \leq n$. By the equality $f_{n-1}d = df_n$, we have an induced map $f'_n : Z_n(P) \to Z_n(Q)$.
But $Z_n(P) = B_n(P)$ by exactness of $P_\bullet$, so $d : P_{n+1} \twoheadrightarrow P_n$ factorizes by $d : P_{n+1} \to Z_n(P)$ (and similarly for Q). We therefore have, by projectivity of P_{n+1}, the existence of a map f_{n+1} such that the following diagram commutes:

$$\begin{tikzcd}
 & P_{n+1} \arrow[d, two heads, "d"] \arrow[ddl, dashed, "\exists f_{n+1}"'] \\
 & Z_n(P) \arrow[d, "f'_n"] \\
Q_{n+1} \arrow[r, two heads, "d"] & Z_n(Q)
\end{tikzcd}$$

Moreover, $df_{n+1} = f'_n d = f_n d$ so this indeed is a chain map.

To see uniqueness up to homotopy, we let $g_\bullet$ be another candidate to extend f. We want to construct $\{s_n : P_n \to Q_{n+1}\}_{n \geq -1}$ such that $h := f - g = sd + ds$. First define $s_{-1} = 0$. Note that $d_0 h_0 = h_{-1} d_0 = (f - f)d_0 = 0$, so that h_0 factors by $h_0 : P_0 \to Z_0(Q) = B_0(Q)$. By projectivity of P_0, h_0 lifts to a map $s_0 : P_0 \to Q_1$ such that $h_0 = ds_0 = s_{-1}d + ds_0$, and we have the base case.

Assume now by induction that we are given maps $s_i, \forall i < n$ such that $ds_i = h_i - s_{i-1}d$. In particular, the map $h_n - s_{n-1}d : P_n \to Q_n$ satisfies

$$d(h_n - s_{n-1}d) = dh_n - (h_{n-1} - ds_{n-2})d = dh - hd + sdd = 0$$

and this maps factors through $Z_n(Q)$. As before, that means it lifts to a map $s_n : P_n \to Q_{n+1}$ with $ds_n = h_n - s_{n-1}d$, and we have our homotopy by induction. $\square$

Lemma 2.17 (Horseshoe lemma)**.** *Given a commutative diagram*

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
\cdots \arrow[r] & P'_2 \arrow[r] & P'_1 \arrow[r] & P'_0 \arrow[r, "\epsilon'"] & A' \arrow[r] \arrow[d, "\iota_A"] & 0 \\
 & & & & A \arrow[d, "\pi_A"] & \\
\cdots \arrow[r] & P''_2 \arrow[r] & P''_1 \arrow[r] & P''_0 \arrow[r, "\epsilon''"] & A'' \arrow[r] \arrow[d] & 0 \\
 & & & & 0 &
\end{tikzcd}$$

where the column is exact and the rows are projective resolutions, and defining $P_n := P'_n \oplus P''_n$, we have that $P_\bullet$ is a projective resolution of A, and that the right hand column of the diagram lifts to an exact sequence of chain complexes

$$0 \to P' \xrightarrow{\iota} P \xrightarrow{\pi} P'' \to 0$$

where $\iota_n : P'_n \to P_n, \pi_n : P_n \to P''_n$ are the natural inclusion and projection.

Proof. See exercise sheet 5. $\square$